Hwk to Turn in: 2.41, 2.39, 2.42
+ Test Corrections.

2.43

Given condition: constraint

$$xyz = 1000 = \text{Volume}$$

quantity to minimize or maximize
= function.

In this question: heat loss.

$H(x,y,z)$ = total heat loss ← proportional to area.

floor ← $C \cdot xy$ ← factor converting area to heat loss.

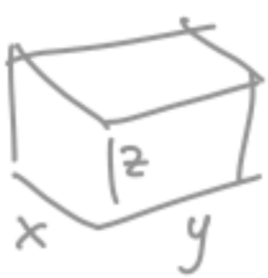
+

each wall (4 of them)

eg. ← eg $(Cyz) \cdot 3$

+

ceiling



$H(x,y,z)$ = sum of those 6 losses

Another optimization question.

We are designing a cardboard box, and the top & bottom of the box require twice as much material. How should the box be designed so that it minimizes the amount of cardboard for a given volume?

fcn. constraint

$g(x,y,z) = xyz = V$ = constant.

also, $x, y, z > 0$.

function cardboard amount $C(x,y,z) = \overset{\text{bottom}}{2xy} + \overset{\text{top}}{2xy} + xz + xz + yz + yz$

$= 4xy + 2xz + 2yz$

Lagrange:

$$\nabla C = \lambda \nabla V, \quad g(x,y,z) = xyz = V$$

$$\Rightarrow (4y+2z,\ 4x+2z,\ 2x+2y) = \lambda(yz, xz, xy)$$

$$\begin{aligned}
4y+2z &= \lambda yz \Rightarrow 4xy + 2xz = \lambda xyz \quad (1)\\
4x+2z &= \lambda xz \Rightarrow 4xy + 2yz = \lambda xyz \quad (2)\\
2x+2y &= \lambda xy \Rightarrow 2xz + 2yz = \lambda xyz \quad (3)\\
xyz &= V
\end{aligned}$$

Subtract (1) − (3) $\Rightarrow$ $2xz - 2yz = 0$

div by $2z$ $\quad x - y = 0 \Rightarrow x = y$.

Subtract (2) − (3) $\Rightarrow$ $4xy - 2xz = 0$

div by $2x$ (nonzero) $\quad 2y - z = 0$

$2y = z$.

Our critical point has

$x = y$, $z = 2y$

$\rightsquigarrow$ box should be twice as tall as wide.

length & width are same.

In terms of V: $xyz = V$ $\quad y(y)(2y) = V$

$2y^3 = V$

$$\Rightarrow \boxed{\begin{aligned} y &= \sqrt[3]{\tfrac{V}{2}} . \\ x &= \sqrt[3]{\tfrac{V}{2}} \\ z = 2y &= 2\sqrt[3]{\tfrac{V}{2}} \end{aligned}}$$

Question: How do we know that this provides minimum of $C(x,y,z)$.

$xyz = V$

$x, y, z > 0$.

level sets for z are

$xy = \frac{V}{z} = \text{cst}$

hyperbola.

Not bounded, but it's closed.

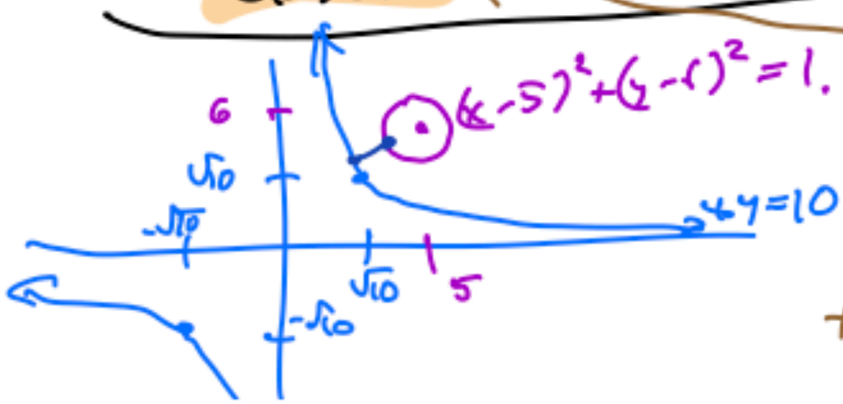
side view

function $C(x,y,z) = 4xy + 2xz + 2yz$

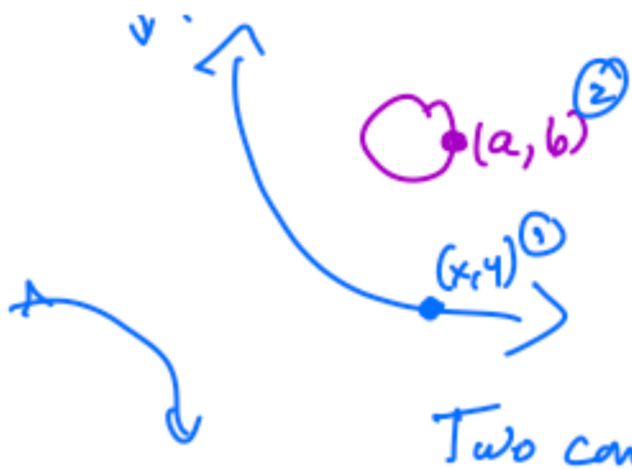
as x, y, or z go to ∞, at least one of these terms will get really huge.

We can surmise that our critical really is a minimum.

Example: Find the points on the curves $xy = 10$ and $(x-5)^2 + (y-6)^2 = 1$ that are the closest.



We want to minimize distance between two points. (actually easier to minimize distance^2)



Function:

$D(x,y,a,b)$

$= (x-a)^2 + (y-b)^2$

↑ we want to minimize.

Two constraints:

1 $xy = 10$

2 $(a-5)^2 + (b-6)^2 = 1$

For more than one constraint, the Lagrange multipliers formula is like this:

For critical pts of $F(x_1, x_2, \dots)$ restricted to

$$g_1(x_1, x_2, \dots) = c_1$$
$$g_2(x_1, x_2, \dots) = c_2$$
$$\vdots$$

Is: $\nabla F = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \cdots$

$$g_1 = c_1$$
$$g_2 = c_2$$
$$\vdots$$

For our example: $D(x,y,a,b) = (x-a)^2 + (y-b)^2$

$g_1 = xy = 10$

$\nabla D = \big(2(x-a), 2(y-b), -2(x-a), -2(y-b)\big)$

$\nabla g_1(x,y,a,b) = (y, x, 0, 0)$

$g_2 = (a-5)^2 + (b-6)^2 = 1$ $\quad \nabla g_2 = (0, 0, 2(a-5), 2(b-6))$

$$\nabla D = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$2(x-a) = \lambda_1 y + \lambda_2 0$$
$$2(y-b) = \lambda_1 x + \lambda_2 0$$
$$-2(x-a) = \lambda_1 0 + \lambda_2 (2(a-5))$$
$$-2(y-b) = \lambda_1 0 + \lambda_2 (2(b-6))$$
$$xy = 10$$
$$(a-5)^2 + (b-6)^2 = 1$$

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simplified

$$2(x-a) = \lambda_1 y$$
$$2(y-b) = \lambda_1 x$$
$$-2(x-a) = 2\lambda_2 (a-5)$$
$$-2(y-b) = 2\lambda_2 (b-6)$$
$$xy = 10$$
$$(a-5)^2 + (b-6)^2 = 1.$$

$\rightarrow$ Sagemath.

see solution.

$[x = (-3.298877731836976), y = (-3.031333333333333), a = 4.323383084577115, b = 5.263664962642548]$

$[x = 2.127860262008733, y = 4.699554896142433, a = 4.089028776978417, b = 5.587530966143682]$

$[x = 2.127861089187056, y = 4.699554896142433, a = 5.910971223021583, b = 6.412469033856318]$

$[x = (-3.298877731836976), y = (-3.031333333333333), a = 5.676616915422885, b = 6.736334405144695]$

We see that $(x,y) = (2.128, 4.700)$ and $(a,b) = (4.089, 5.588)$ are the two closest points!